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Polynomial Division Template

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Abstract—*The compact template for the division of two univariate polynomials to find the quotient and reminder is derived. The process is very simple, efficient and direct, comparing to the familiar classical long polynomial division and synthetic polynomial division***.**
Keywords — *Polynomial division; Long polynomial division; Synthetic polynomial division.*

I. INTRODUCTION

There are several approaches for finding the quotient and remainder from dividing two given univariate polynomials. Long polynomial division is very popular but tedious in computation, and widely used even by high school students. Synthetic polynomial division is fairly easy to use but only appropriate for the linear divisor [1]. Convolution polynomial division [2] is direct in operation, and used in MATLAB built-in routine.

This work presents a compact template for polynomial division. The process is very simple and straightforward and does not need to write down any intermediate steps, as in the familiar classical long polynomial division and synthetic polynomial division. It is extremely suitable for hand computation with a plain calculator.

Typical numerical examples are provided to show the merit of the approach presented.

II. FORMULATION

The division of two given polynomials, dividend $b(x)$ of degree n and divisor $a(x)$ of degree m, to get the resulted polynomials, quotient $q(x)$ of degree n-m and remainder $r(x)$ of degree m-1, may be expressed as

$$\frac{b(x)}{a(x)} = q(x) + \frac{r(x)}{a(x)}$$

or $$b(x) = a(x) \cdot q(x) + r(x)$$

where

$$b(x) = b_0 x^n + b_1 x^{n-1} + \cdots\cdots + b_{n-1} x + b_n$$
$$a(x) = a_0 x^m + a_1 x^{m-1} + \cdots\cdots + a_{m-1} x + a_m$$

$$q(x) = q_0 x^{n-m} + q_1 x^{n-m-1} + \cdots\cdots + q_{n-m-1} x + q_{n-m}$$
$$r(x) = r_0 x^{m-1} + r_1 x^{m-2} + \cdots\cdots + r_{m-2} x + r_{m-1}$$

Then the coefficients of x^ℓ in both side of the equation after substituting of the expansion forms of $b(x)$, $a(x)$ and $q(x)$, $r(x)$ will give the following relation:

$$b_\ell = a_\ell q_0 + a_{\ell-1} q_1 + \cdots\cdots + a_1 q_{\ell-1} + a_0 q_\ell + r_{\ell-(n-m+1)}, \qquad \ell = 0, 1, \cdots\cdots, n$$

where it is understood that

$$b_\ell = 0,\ \ell > n, \quad a_\ell = 0,\ \ell > m, \quad \text{and} \quad q_\ell = 0,\ \ell > n-m, \quad r_\ell = 0,\ \ell < 0.$$

From the relation the polynomial division manipulation may be conveniently cast into the following templates:

$$\begin{array}{c|ccccccccc}
 & b_0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & b_n \\
\div) & a_0 & \cdots & \cdots & \cdots & \cdots & a_m & & & \\
\hline
 & q_0 & \cdots & \cdots & q_{n-m} & & & & & \\
 & & & & & r_0 & \cdots & \cdots & \cdots & r_{m-1}
\end{array} \qquad \text{for } \; n-m<m$$

or

$$\begin{array}{c|ccccccccc}
 & b_0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & b_n \\
\div) & a_0 & \cdots & \cdots & a_m & & & & & \\
\hline
 & q_0 & \cdots & \cdots & \cdots & \cdots & q_{n-m} & & & \\
 & & & & & & & r_0 & \cdots & r_{m-1}
\end{array} \qquad \text{for } \; n-m>m$$

It follows that the desired coefficients are thus determined:

$$q_k = \Big(b_k - \sum_{\ell=\max(0,k-m)}^{k-1} a_{k-l}\, q_l \Big) \Big/ a_0 \,, \qquad k = 0, \cdots\cdots, n-m$$

$$r_{k-(n-m+1)} = \Big(b_k - \sum_{\ell=\max(0,k-m)}^{n-m} a_{k-l}\, q_l \Big) \,, \qquad k = n-m+1, \cdots\cdots, n$$

The total number of multiplication/division arithmetic operations for this approach is found to be merely $m \cdot (n-m)$.

In practical computation to save the space, we may combine the last two lines into a single line in the polynomial division template. For illustration, the compact templates for (n, m) = (8, 5) and (n, m) = (8, 3) are as shown:

$$\begin{array}{c|cccc|ccccc}
 & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 \\
\div) & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & & & \\
\hline
 & q_0 & q_1 & q_2 & q_3 & r_0 & r_1 & r_2 & r_3 & r_4
\end{array} \qquad \text{for } \; n-m<m$$

$$\begin{aligned}
q_0 &= (\; b_0 \;) / a_0 \\
q_1 &= (\; b_1 \; - a_1 q_0) / a_0 \\
q_2 &= (\; b_2 \; - a_2 q_0 \; - a_1 q_1 \;) / a_0 \\
q_3 &= (\; b_3 \; - a_3 q_0 \; - a_2 q_1 \; - a_1 q_2 \;) / a_0 \\
r_0 &= (\; b_4 \; - a_4 q_0 \; - a_3 q_1 \; - a_2 q_2 \; - a_1 q_3 \;) \\
r_1 &= (\; b_5 \; - a_5 q_0 \; - a_4 q_1 \; - a_3 q_2 \; - a_2 q_3 \;) \\
r_2 &= (\; b_6 \; - a_5 q_1 \; - a_4 q_2 \; - a_3 q_3 \;) \\
r_3 &= (\; b_7 \; - a_5 q_2 \; - a_4 q_3 \;) \\
r_4 &= (\; b_8 \; - a_5 q_3 \;)
\end{aligned}$$

And

$$\begin{array}{c|cccccc|ccc}
 & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 \\
\div) & a_0 & a_1 & a_2 & a_3 & & & & & \\
\hline
 & q_0 & q_1 & q_2 & q_3 & q_4 & q_5 & r_0 & r_1 & r_2
\end{array} \qquad \text{for } \; n-m>m$$

$$\begin{aligned}
q_0 &= (\; b_0 \;) / a_0 \\
q_1 &= (\; b_1 \; - a_1 q_0) / a_0 \\
q_2 &= (\; b_2 \; - a_2 q_0 \; - a_1 q_1 \;) / a_0 \\
q_3 &= (\; b_3 \; - a_3 q_0 \; - a_2 q_1 \; - a_1 q_2 \;) / a_0 \\
q_4 &= (\; b_4 \; - a_3 q_1 \; - a_2 q_2 \; - a_1 q_3 \;) / a_0 \\
q_5 &= (\; b_5 \; - a_3 q_2 \; - a_2 q_3 \; - a_1 q_4) / a_0 \\
r_0 &= (\; b_6 \; - a_3 q_3 \; - a_2 q_4 \; - a_1 q_5 \;) \\
r_1 &= (\; b_7 \; - a_3 q_4 \; - a_2 q_5 \;) \\
r_2 &= (\; b_8 \; - a_3 q_5 \;)
\end{aligned}$$

Typical numerical examples for the cases $m - n < m$ and $m - n > m$ are presented below to show the merits of the approach derived.

<u>*Example* 1.</u> For $m - n < m$,

Given: $b(x) = 4x^8 + 5x^7 - x^6 + 7x^5 - 6x^4 + x^3 + 2x^2 - 3x + 7$ and $a(x) = 3x^5 + x^4 - 7x^3 + 5x^2 - 4x + 2$

yields: $q(x) = \frac{4}{3}x^3 + \frac{11}{9}x^2 + \frac{64}{27}x + \frac{176}{81}$ and $r(x) = \frac{619}{81}x^4 + \frac{533}{81}x^3 - \frac{148}{81}x^2 + \frac{77}{81}x + \frac{215}{81}$

since

$$\begin{array}{c|cccc|ccccc}
 & +4 & +5 & -1 & +7 & -6 & +1 & +2 & -3 & +7 \\
\div) & +3 & +1 & -7 & +5 & -4 & +2 & & & \\
\hline
 & +\frac{4}{3} & +\frac{11}{9} & +\frac{64}{27} & +\frac{176}{81} & +\frac{619}{81} & +\frac{533}{81} & -\frac{148}{81} & +\frac{77}{81} & +\frac{215}{81}
\end{array}$$

<u>*Example* 1.</u> For $m - n > m$,

Given: $b(x) = 4x^8 + 5x^7 - x^6 + 7x^5 - 6x^4 + x^3 + 2x^2 - 3x + 7$ and $a(x) = 3x^3 + x^2 - 7x + 5$

yields: $q(x) = \frac{4}{3}x^5 + \frac{11}{9}x^4 + \frac{64}{27}x^3 + \frac{176}{81}x^2 + \frac{187}{243}x + \frac{872}{729}$ and $r(x) = -\frac{3407}{729}x^2 + \frac{1112}{729}x + \frac{743}{729}$

since

$$\begin{array}{c|cccccc|ccc}
 & +4 & +5 & -1 & +7 & -6 & +1 & +2 & -3 & +7 \\
\div) & +3 & +1 & -7 & +5 & & & & & \\
\hline
 & +\frac{4}{3} & +\frac{11}{9} & +\frac{64}{27} & +\frac{176}{81} & +\frac{187}{243} & +\frac{872}{729} & -\frac{3407}{729} & +\frac{1112}{729} & +\frac{743}{729}
\end{array}$$

It is noted that for the case of $m - n = m$, the quotient becomes simply a constant.

III. COMPUTER ROUTINE IN MATLAB

A simple computer routine in MATLAB is presented. Inputs `b` and `a`, and outputs `q` and `r` are the coefficient vectors of given dividend $b(x)$ and divisor $a(x)$, and resulted quotient $q(x)$ and reminder $r(x)$, respectively

```
function [q,r] = poly_div(b,a)
%   Polynomial division by template
%   F C Chang       10/25/2012
%
       n = length(b)-1;  m = length(a)-1;
    if m > n,  q = 0;  r = b;   return;  end;
       a = [a,zeros(1,n-m)];    q = 0;
  for  k = 1:n+1,
    if k < n-m+2,
       q(k) = (b(k)-[q(1:k-1)]*[a(k:-1:2)].')/a(1);
    else
       r(k-(n-m+1)) = b(k)-[q(1:n-m+1)]*[a(k:-1:k-n+m)].';
    end
  end;
```

It is noted that the current routine `[q,r] = poly_div(b,a)` is similar to the MATLAB built-in routine `[q,r] = deconv(b,r).`

IV. CONCLUSION

The useful template is derived for division of polynomials. By comparison with other methods, this approach is simple and effective. The desired quotient and reminder are directly determined without writing down any intermediate data as in the familiar classical longhand polynomial division and synthetic polynomial division.

One of the important applications is to find the roots with multiplicities of any given polynomial after the GCD of the polynomial and its derivative is computed [3] [4].

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REFERENCES

[1] L. Zhou, Short division of polynomials, *The College Mathematics Journal*, *40*, 2009, 44-46.
[2] F.C. Chang, Polynomial division by convolution, *Applied Mathematics E-note*, *11*, 2011, 249-254.
[3] F.C. Chang, Solving multiple-roots polynomials, *IEEE Antennas & Propagation Magazine*, *51(6)*, 2009, 151-155.
[4] Z. Zeng, Computing multiple roots of inexact polynomials, *Math. Comput.*, *74*, 2005, 869-903.